MATH299G

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Contents

Chapter 1

Introduction

1.1 Logistics

Main reference: "Combinatorial Game Theory" by Aaron Siegel

1.2 Introduction

This is a class on Combinatorial Game Theory, where we develop the theory behind games, nimbers, and more.

Definition 1 (Short Combinatorial Game)**.** A game is a short combinatorial game if it satisfies all of the following properties:

- The game has exactly two players.
- The game has some number of **positions**, which can be moved from by a player to other positions.
- The two players move in sequence.
- The game has no randomness.
- Both players have complete information about the game state.
- The game must end at some point. At some point a player will be left unable to move. (There are no loops).

Later, that last rule can be lifted, but for now, we will only focus on short combinatorial games.

Example (Hackenbush)**.** Hackenbush is a combinatorial game! (See the following definition to see that it satisfies the definition.)

Example (Monopoly (nonexample)). Note that there is randomness in Monopoly, therefore it is not combinatorial.

Example (Tic-Tac-Toe (nonexample)). The existence of draws (there could be no win / loss).

Example (Poker (nonexample)). Hiding of your cards (incomplete information).

Chapter 2

Hackenbush and Game States

2.1 Hackenbush

For starters, we introduce a game known as **Blue-Red Hackenbush**.

Definition 2 (Blue-Red Hackenbush). We have two players, **Left** and **Right**. Left moves by deleting any blue edge together with any other edges which are no longer connected to the ground. Right moves by deleting any red edge together with any other edges which are no longer connected to the ground. The loser is the player who cannot play a move.

Example (Example Hackenbush Game). This is a simple example of a Hackenbush game.

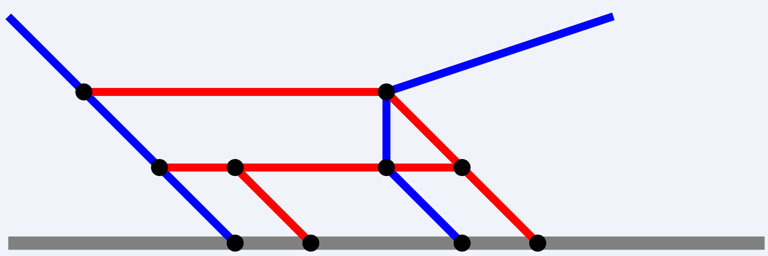

Figure 2.1: Hackenbush Game

Who wins in this scenario? We shall see.

Blue-Red Hackenbush is an example of a combinatorial game. What we want to do with this game is look at a specific position and see whether Left or Right has a winning strategy from that position.

First lets take a look at some simple positions.

Example (Simplest Possible Position). This is the simplest possible position.

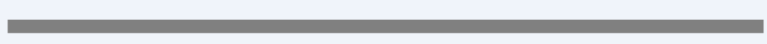

Figure 2.2: Simplest Possible Hackenbush Position

We can see here that both Left and Right have no moves. Therefore, whoever moves first will always lose, so the winner is **whoever moves second**.

Example (Second Simplest Position). This is the second simplest possible position.

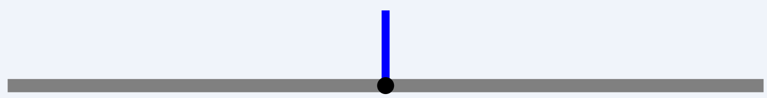

Figure 2.3: Second Simplest Position

Here we see that if Right moves first, they have no moves and so they lose. However, if Left moves first, they can get rid of the branch, and Right still loses. This is **always a win for Left**.

There are a few ways to represent a combinatorial game:

- As a single letter (eg G), for when we don't know much about the game.
- As a colored directed graph, representing options for Left and Right, with a final state.
- A set of options for left and right, represented with a set split into two. For example, $\left\{0\middle|\right\}$. We will mainly be using this and the single letters.

Remark. Before discussing positions, let us note that we are only discussing **short** or **finite loopfree** combinatorial games (for now). These games must end in the zero position after a finite number of moves.

2.2 Positions

Definition 3 (Position). A **position** is defined in terms of the positions to which each player can move to. We notate a position using a pair of braces and a vertical bar. The moves Left can make go to the left of the bar, and the moves right can make go on the Right.

Example (Simplest Position).

$$0 = \left\{\middle|\right\}$$

Here neither player has any moves, meaning this position is the empty board.

Example (Second Simplest Position). Well now that we have 0, we can say for example that Left can do a move which changes the position to 0. Therefore, we get (for the second simplest position)

$$\left\{ 0 \middle| \right\}$$

We denote that by 1. In this case, note that Left always wins.

We can actually construct 4 different positions, each corresponding to a different **outcome classes**. Note that

- $0 = \left\{ \middle| \right\}$, the second player always wins (can be notated as $\mathcal{P}$ for previous).
- $1 = \left\{ 0 \middle| \right\}$, the Left player always wins (can be notated as $\mathcal{L}$ for Left).
- $-1 = \left\{ \middle| 0 \right\}$, the Right player always wins (can be notated as $\mathcal{R}$ for Right).
- $\star = \left\{ 0 \middle| 0 \right\}$, **fuzzy**, the first player always wins (can be notated as $\mathcal{N}$ for next).

Definition 4 (Comparison Notation). For any position G, we introduce the following notation.

- $G = 0$ means that G is zero, or the second player can always win.
- $G > 0$ means that G is positive, or the Left player can always win.
- $G < 0$ means that G is negative, or the Right player can always win.
- $G \mid\mid 0$ means that G is fuzzy, or the first player can always win.

These can also be combined, for example $G \geq 0$ means G is positive or zero and $G \mid\mid> 0$ means that G is positive or fuzzy.

Definition 5 (Outcome Classes). For any position G, $o(G)$ is defined to be the outcome class of G.

Example (Outcome classes). Four examples of games that lie in outcome classes:

- $o(0) = \mathcal{P}$
- $o(1) = \mathcal{L}$
- $o(-1) = \mathcal{R}$
- $o(\star) = \mathcal{N}$

Remark. Note that the fuzzy state does not actually correspond with a state in our current Hackenbush definition. We introduce the concept of a **green stalk** to allow fuzzy games. Either player can remove the green stalk. Therefore, we can now construct $\star = \left\{0\middle|0\right\}$ as the following:

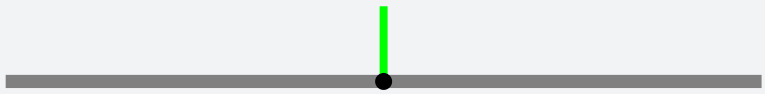

Figure 2.4: Star Position: First player wins

Note. We can also arrange these outcome classes into a table:

Table 2.1: Outcome classes

	Left starts and wins	Left starts and loses
Right starts and wins	$G \mid\mid 0$	$G < 0$
Right starts and loses	$G > 0$	$G = 0$

Theorem 1 (Fundamental theorem of CGT)**.** Let G be a short combinatorial game. Then either Left can win by moving first, or else Right can win by moving second, but not both.

For this proof, we need some machinery. Therefore, we introduce a lemma and a definition below.

Lemma 1. Let $S(n)$ and $T(n)$ be statements that depend on n. Then if

- $S(0)$ and $T(0)$ are true
- $S(n) \implies T(n+1)$ and $T(n) \implies S(n+1)$ for all $n \geq 0$

Then $S(n)$ and $T(n)$ are true for all $n \geq 0$.

Definition 6 (Formal Birthday)**.** We define the formal birthday of a game G, denoted $\tilde{b}(G)$ to be the length of the longest path to the zero position from the initial position:

$$\tilde{b}(G) = \max\left(\tilde{b}(G^L), \tilde{b}(G^R)\right) + 1$$

where $\tilde{b}(0) = 0$.

Proof 1. (Proof of the Fundamental Theorem of CGT).

Note some properties of the formal birthday:

- $\tilde{b}(0) = 0$
- If $\tilde{b}(G) = 0$, then $G = 0$.

We proceed for the proof by induction.

Let $S(n)$ be the statement "For a short game with formal birthday $\tilde{b}(G) \leq n$, either Left wins by playing first, or Right wins by playing second, but not both."

Let $T(n)$ be the statement "For a short game with formal birthday $\tilde{b}(G) \leq n$, either Right wins by playing first, or Left wins by playing second, but not both."

Base case: Suppose $\tilde{b}(G) \leq 0$. Then G is the zero position, so whoever moves second wins. Therefore, for $S(0)$, Right wins, and $T(0)$ Left wins.

Assume $S(k)$ and $T(k)$ are true for arbitrary k. We show $S(k+1)$ is true. ($T(k+1)$ is an exercise).

We see that $S(k+1)$ states that if $\tilde{b}(G) \leq k+1$, then either Left wins by going first or Right wins by going second. Therefore, let us look at the left options of our game G.

For all the left options G^L of G, either right wins on G^L going first, or left wins on G^L going second. Either there is such a G^L in which Left wins going second (so L will move to G^L), or there is not, and in the second case Right wins going second on G no matter Left's first move. Therefore, $S(k+1)$ is true as well. $T(k+1)$ is left as exercise. □

Theorem 2 (Position Classification)**.** Every position is either zero, positive, negative, or fuzzy.

Proof 2. We proceed by induction. In the base case, we have $G = \left\{ \middle| \right\} = 0$.

Suppose for some $G = \left\{ G^L \middle| G^R \right\}$, that all G^L and all G^R are either positive, negative, zero, or fuzzy.

Suppose without loss of generality that Left starts the game. Then we consider all of Left's options, in G^L. If there exists a position $P \in G^L \geq 0$, then Left can move to that position. Since it is greater than or equal to zero, Right has no winning strategy. Therefore, since Left has a winning strategy, then G is either positive or fuzzy.

If $P \notin G^L \geq 0$, then Right has a winning strategy after Left's move. This means that $G \leq 0$, as Right can always follow the strategy for the new position after Left plays. □

2.3 Arithmetic

Currently, we are able to compare our games to the 0 game. However, what if we want to compare different games? We can do this by introducing arithmetic.

Definition 7 (Sum of two positions)**.** The sum of two positions is defined as on a player's turn, they can choose to **either** play a move in the first position, or play a move in the second position. They **must** move in exactly one component. This can be notated as following:

$$G + H = \left\{G^L \middle| G^R\right\} + \left\{H^L \middle| H^R\right\} = \left\{G^L + H, G + H^L \middle| G^R + H, G + H^R\right\}$$

(Note that when we say $G^R + H$, what we really mean is take every option on the right side of G and add each of them to H)

Definition 8 (Negation of a position)**.** The negation of a position is defined as "swapping" the roles of each player. Left now plays as Right would, and vice versa. In the notation:

$$-G = -\left\{G^L \middle| G^R\right\} = \left\{-G^R \middle| -G^L\right\}$$

The reason we negate after swapping is because we need the position swap to persist through all turns, and not where they swap for one turn before reverting back.

Definition 9 (Subtraction)**.** We define

$$G - H = G + (-H)$$

Example (Simple addition)**.**

$$\begin{aligned} 1 + 1 &= \left\{0 \middle| \right\} + \left\{0 \middle| \right\} \\ &= \left\{0 + 1, 0 + 1 \middle| \right\} \\ &= \left\{1, 1 \middle| \right\} \text{ (proof shown below.)} \end{aligned}$$

Here we can do a small bit of simplification. Note that left can either move to 1 or 1. Since it has no unique move, we can just remove the duplicate, giving us $1 + 1 = \left\{1 \middle| \right\}$. We can give this position a name, namely 2.

Proposition 1. $\forall G,\ G + 0 = G$

Proof 3. We proceed by induction. We can see that

$$0 + 0 = \left\{0^L + 0, 0 + 0^L \middle| 0^R + 0, 0 + 0^R\right\}$$

Since both 0^L and 0^R are empty, there are no additions and we get $\left\{ \middle| \right\} = 0$. So $0 + 0 = 0$.

We assume that for $G = \left\{G^L \middle| G^R\right\}$, for any position $P \in G^L \cup G^R$, we have that $P + 0 = P$. Then we see that

$$G + 0 = \left\{G^L + 0, G + 0^L \middle| G^R + 0, G + 0^R\right\}$$

Since 0^L and 0^R are both empty, we get

$$G + 0 = \left\{G^L + 0 \middle| G^R + 0\right\}$$

By induction, this is the same as $\left\{G^L \middle| G^R\right\} = G$. □

Proposition 2. $\forall G, H,\ G + H = H + G$

Intuitively, we see that this is true as the order in which you combine positions does not matter, since you are free to move in any component.

Proposition 3. $\forall A, B, C,\ (A + B) + C = A + (B + C)$

Same intuition as commutativity, however these proofs are left as an exercise.

Proposition 4. $-(G + H) = -G - H$

Since a sum is just allowing a player to move in one of many components, swapping the roles of Left and Right in a sum is done by swapping their roles in each component and adding them together.

2.4 Comparisons

Since we now have the concept of addition, we can use it to compare arbitrary positions.

Definition 10 (Comparisons)**.** We introduce the concept of comparing two nonzero games.

- $G > H$ if $G - H$ is positive
- $G < H$ if $G - H$ is negative
- $G = H$ if $G - H$ is zero
- $G \mid\mid H$ if $G - H$ is fuzzy (G is confused with H)

Remark. Please note here that the $=$ symbol now does not mean exactly the same, it means now the relationship above. For example, previously $\left\{ \middle| \right\}$ was not equal to $\left\{-1 \middle| 1\right\}$, but in this case it now is. (Verify this as an exercise). We now use the $\equiv$ to denote identical positions.

Proposition 5. $\forall G,\ G = G$

Intuitively, if we can see that $G + -G = 0$. This is because if we combine a position G with $-G$, every time Left makes a move (in either of these games), Right can make a "copy" move on the other board taking off the exact same stalk. Therefore, in the end Right will make the last move.

Proposition 6. If $G = H$, then $H = G$.

Proof 4. If $G = H$, then by definition $G - H = 0$. Consider the negative position, $-(G - H) = -G + H = H - G$. If this position is a second player win, then $H = G$.

However, this is the negation of a zero position. Left starts and loses in $G - H$, so Right starts and loses in $H - G$. Right starts and loses in $G - H$, so Left starts and loses in $H - G$. Since both players lose if they start, $H - G = 0$ so $H = G$. □

Proposition 7. If $A = B$ and $B = C$, then $A = C$.

Proof 5. Since $A = B$, $A - B = 0$. Since $B = C$, $B - C = 0$. Then

$$A - C = A - B + B - C = 0 + 0 = 0$$

Therefore, $A = C$. □

Remark. We have just shown that equality of games forms an equivalence relation. We call the equivalence class that a given position falls into its "value". If we must distinguish between exact positions, we call it its "form", and use the $\equiv$ operator. The names $0, 1, -1, \star, 2$ are all names we give to equivalence classes of positions.

Remark. Note that the set of games forms a group under the addition operation.

2.5 Numbers and Notating Positions

Now we have a formal way to describe an arbitrary position in any combinatorial game. How can we use this? Previously, we have seen the following example show up:

Example (2).

$$2 = 1 + 1 = \left\{1, 1 \middle| \right\} = \left\{1 \middle| \right\}$$

However, what if we look at two branches on top of each other instead of next to each other? Then we are not exactly adding the two branches together. We instead get $\left\{0, 1 \middle| \right\}$. Is this the same as 2?

Theorem 3 (Dominant Positions)**.** For some position

$$G = \left\{A, B, C, \dots \middle| X, Y, Z, \dots\right\}$$

If $B \geq A$, then G is equal to the position

$$H = \left\{B, C, \dots \middle| X, Y, Z, \dots\right\}$$

Likewise, if $Y \leq X$, then G is equal to

$$H' = \left\{B, C, \dots \middle| Y, Z, \dots\right\}$$

Proof 6. (For left only).

We wish to show that $H = G$. We can show this by showing that $H - G = 0$, or that the second player can always win $H - G$.

Recall that

$$-H = \left\{-X, -Y, -Z, \dots \middle| -B, -C, \dots\right\}$$

With one exception, note that the first player's move can be copies onto the other component by the second player. For example, if the first player plays to position P, the second player can play to position $-P$ in the other game, which always leaves us with a position of $P - P = 0$, so the second player wins.

For the exception, suppose Left is first. If they move G to A, then now Right cannot move to $-A$. However, they can move to $-B$ in H. Therefore, the entire position is now $A - B$. By the premise, $B \geq A$. Therefore, $A - B \leq 0$, so Right always has a winning strategy.

In either case, the second player has a winning strategy, so the difference is zero and $G = H$. □

Definition 11 (Notation of Numbers)**.** We define the numbers to be the following:

$$1 = \left\{0 \middle| \right\}$$
$$2 = \left\{1 \middle| \right\}$$
$$3 = \left\{2 \middle| \right\}$$
$$\dots$$

For any natural number n, we define

$$n = \left\{n - 1 \middle| \right\}$$

We can also define the negative integers similarly, namely

$$-n = \left\{ \middle| -(n-1) \right\}$$

Now what if we instead allow multiple colors to be stacked on top of each other? For example, a red on top of a blue. Note that this position is the following:

$$? = \left\{ 0 \middle| 1 \right\}$$

What exactly is this position? Well, we can first note that $? > 0$. This is because we can see that right will always lose. We can also observe an interesting fact about it. Consider the following game.

$$\left\{ 0 \middle| 1 \right\} + \left\{ 0 \middle| 1 \right\} + \left\{ \middle| 0 \right\}$$

Lets see what happens when we play this game.

- **Left plays first:**
  If Left plays first, there is only one distinct move Left can make. Therefore, we get

  $$\left\{ 0 \middle| 1 \right\} + \left\{ \middle| \right\} + \left\{ \middle| 0 \right\}$$

  Then right can just move to

  $$\left\{ 0 \middle| \right\} + \left\{ \middle| \right\} + \left\{ \middle| 0 \right\}$$

  And now note that from our previous definitions, this is just

  $$1 - 1 = 0$$

  So the second player always wins here, meaning right wins.

- **Right plays first:**
  If right moves in the last game, we get

  $$\left\{ 0 \middle| 1 \right\} + \left\{ 0 \middle| 1 \right\}$$

  Then left moves, giving

  $$\left\{ \middle| \right\} + \left\{ 0 \middle| 1 \right\}$$

  Now since we showed that $\left\{ 0 \middle| 1 \right\} > 0$, we see that left must win here.
  If right moves in the other game, we get

  $$\left\{ 0 \middle| 1 \right\} + \left\{ 0 \middle| \right\} + \left\{ \middle| 0 \right\}$$

  Now note that left can just play in the first game, making the state

  $$\left\{ 0 \middle| \right\} + \left\{ \middle| 0 \right\}$$

  Since it is Right's move and we showed this state is $1 - 1 = 0$, then the second player will win (left).

So wait! This game is equal to 0. This means that

$$\begin{aligned} x + x + (-1) &= 0 \\ 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

This game is actually $\frac{1}{2}$.

We can actually repeat this by adding another branch to our stack, which gives us

$$\left\{0 \middle| 1, \frac{1}{2}\right\} = \left\{0 \middle| \frac{1}{2}\right\} = \frac{1}{4}$$

Definition 12 (Fractional Powers of 2)**.** We define the sequence of positions

$$G_0 = 1 = \left\{0 \middle| \right\}$$

$$G_n = \left\{0 \middle| G_{n-1}\right\}$$

Theorem 4 (Fractional Powers of 2)**.**

$$G_n = \frac{1}{2^n}$$

Proof 7. We want to show that $G_n + G_n = G_{n-1}$. We proceed by induction.

We have already shown that $\left\{0 \middle| 1\right\} + \left\{0 \middle| 1\right\} - \left\{0 \middle| \right\} = 0$, so

$$G_1 + G_1 = G_0$$

Suppose $G_{n-1} + G_{n-1} = G_{n-2}$. We wish to show $G_n + G_n = G_{n-1}$. We can consider

$$G_n + G_n - G_{n-1} = \left\{0 \middle| G_{n-1}\right\} + \left\{0 \middle| G_{n-1}\right\} + \left\{-G_{n-2} \middle| 0\right\}$$

and show that this is a second player win, equalling 0.

- Suppose Left starts:
  - If Left moves $\left\{0 \middle| G_{n-1}\right\} \to 0$, Right can move the other $\left\{0 \middle| G_{n-1}\right\} \to G_{n-1}$. This leaves Left with a position of $G_{n-1} - G_{n-1} = 0$, so Right wins.
  - If Left moves $\left\{-G_{n-2} \middle| 0\right\} \to -G_{n-2}$, Right can move $\left\{0 \middle| G_{n-1}\right\} \to G_{n-1}$. This leaves Left with the position $G_n + G_{n-1} - G_{n-2}$. By induction, $G_{n-1} + G_{n-1} = G_{n-2}$,

so $G_{n-1} - G_{n-2} = -G_{n-1}$. This means that we have a state with the value $G_n - G_{n-1} < 0$, so Right wins.

- Suppose Right starts:
  - If Right moves to $\left\{0 \middle| G_{n-1}\right\} \to G_{n-1}$, Left can move the other $\left\{0 \middle| G_{n-1}\right\} \to 0$. This leaves Right in the position $G_{n-1} - G_{n-1} = 0$, so Left wins.
  - If Right moves $\left\{-G_{n-2} \middle| 0\right\} \to 0$, Left can move $\left\{0 \middle| G_{n-1}\right\} \to 0$, leaving the sum as only G_n, which is positive, meaning Left wins.

□

Now that we have all powers of two, we can use elementary operations to construct all Dyadic rationals (rational integers which can be expressed as $\frac{p}{2^q}$). We notate this as $\mathbb{D}$.

Theorem 5 (Dyadic Rationals as Games)**.** For any integer p and any natural number m,

$$\left\{\frac{p}{2^m} \middle| \frac{p+1}{2^m}\right\} = \frac{2p+1}{2^{m+1}}$$

Proof 8. Suppose that $2p + 1 > 0$. Then we can write

$$\frac{2p+1}{2^{m+1}} = \left\{0 \middle| \frac{1}{2^m}\right\} + \ldots \text{ (2p + 1 times)}$$

Now let us consider each players options in the right hand side.

If Left starts, their only option is to move one of the $\left\{0 \middle| \frac{1}{2^m}\right\} \to 0$ leaving the final sum as $2p$ copies of $\frac{1}{2^{m+1}}$ plus zero, or $\frac{2p}{2^{m+1}}$.

If Right starts, their only option is to move one of the $\left\{0 \middle| \frac{1}{2^m}\right\} \to \frac{1}{2^m}$, leaving the final sum as $2p$ copies of $\frac{1}{2^{m+1}}$ plus $\frac{1}{2^m}$ which is equal to $\frac{2p}{2^{m+1}} + \frac{1}{2^m}$.

Thus, the right hand side is equal to

$$\left\{\frac{2p}{2^{m+1}} \middle| \frac{2p}{2^{m+1}} + \frac{1}{2^m}\right\}$$

which after simplifying and summing fractions, is also equal to

$$\left\{\frac{p}{2^m} \middle| \frac{p+1}{2^m}\right\}$$

The case where $2p + 1 < 0$ is similar and left as an exercise. □

Cool! So we've constructed the entire set of dyadic rationals $\mathbb{D}$.

Definition 13 (Number). For a position $\left\{G^L \middle| G^R\right\}$ to be called a number, it must satisfy the following:

- All G^L and G^R must be numbers.
- Every G^L must be strictly less than every G^R.

Example (Non example). Note that star is not a number. Namely

$$\star = \left\{0 \middle| 0\right\}$$

Theorem 6 (Simplicity Rule). Suppose we have a number

$$G = \left\{A, B, C, \dots \middle| X, Y, Z, \dots\right\}$$

Where all G^L and G^R are numbers. If there exists a number r such that

$$r > A, r > B, r > C, \dots$$

and

$$r < X, r < Y, r < Z, \dots$$

then G is equal to the simplest r.

Proof 9. See (here).

Recall that any dyadic rational $r \in \mathbb{D}$ can be written in one of these ways:

- $r = \left\{ \middle| \right\}$
- $r = \left\{s \middle| \right\}$, where s is a number simpler than r, and $s < r$.
- $r = \left\{ \middle| s\right\}$, where s is a number simpler than r, and $s > r$.
- $r = \left\{s \middle| t\right\}$, where s and t are numbers simpler than r, and $s < r < t$.

Now for the proof. We must show three things:

1. There exists a simplest r that satisfies the inequalities.
2. $G = r$.
3. r is unique.

Let us begin the proof.

1. Consider the set of all dyadic rationals that satisfy the inequalities. Since the denominators are all finite, there is at least one $r \in \mathbb{D}$ with smallest denominator. If that smallest denominator is 1, choose the one with the smallest absolute value.

2. We wish to show that $G = r$ which will be done by showing that the second player wins the difference $G - r = G + (-r)$.

   Without loss of generality, suppose Left starts. Since any left option is less than r, any of Left's moves in the G component will leave the total position negative, so Right wins.

   Suppose instead that Left moves in $-r$. By our recollection above, we can assume the move to be $-r \to -n$, where n is simpler than r and $r < n$. It follows from our choice of r that either there exists some left option $a \geq r \in G$ or there is a right option $x \leq n \in G$.

   However, we also know $n > r > a$, so we know there must be some right option $x \leq n \in G$. Right can take this option, leaving the final position at $x - n \leq 0$. Since the position is less than or equal to zero, and it is Left's turn, Right wins.

3. If r were nonunique, then by step 2 G would be equal to two different numbers, which is impossible.

□

Chapter 3

Introduction to Impartial Games

3.1 Star and Nim

Definition 14 (Star). Recall in the previous section we looked at a single green branch:

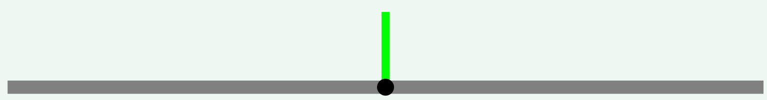

Figure 3.1: Star Position: First player wins

We see that this is $\star = \left\{0 \middle| 0\right\}$. We know from previously that this is always a win for the first player. We can see that $\star \mid\mid 0$.

Definition 15 (Nim). The game Nim consists of a number of piles of stones. Each player on their turn can take a number of stones from any pile. The player who cannot take any stones wins. Note that this is the same as Hackenbush only using green stalks.

Definition 16 (Impartial). A position $G = \left\{G^L \middle| G^R\right\}$ is impartial if

- G^L and G^R are impartial.
- $G^L = G^R$

Note that 0 is an impartial game (the simplest one).

Theorem 7 (Bound of star).

$$\star < \frac{1}{2^n}$$

and

$$\star > -\frac{1}{2^n}$$

Proof 10. (First statement). Consider $\star - \frac{1}{2^n}$. This is the same as

$$\star + \left(-\frac{1}{2^n}\right)$$

If Left goes first, then left has to send

$$-\frac{1}{2^n} \to -\frac{1}{2^{n-k}}$$

where $n - k > 0$. Then right can send $\star \to 0$, and now this is a win for right, as we are left with $-\frac{1}{2^{n-k}}$.

If Right goes first, then right can just send $\star \to 0$, making it a win for right. Therefore,

$$\star < \frac{1}{2^n}$$

□

Theorem 8 (Inverse).

$$\star + \star = 0$$

Proof 11. If Right plays first, they must send $\star + \star \to \star$, and so Right will lose. If Left plays first, a similar argument follows. Therefore, the second player wins. □

Theorem 9 (New positions).

$$\star + k = \left\{k \middle| k\right\}$$

Proof 12. Base case. $\star + 0 = \star$. Assume $\star + k = \left\{k \middle| k\right\}$. Consider $\star + k + 1$.

$$\left\{k \middle| k\right\} + \left\{0 \middle| \right\} = \left\{k+1, \star + k \middle| k+1\right\}$$

Note that $k + 1 > \star + k$. This is because $k + 1 - \star - k = 1 - \star = 1 + \star$. We can see that $1 + \star$

is a win for Left, so by the domination theorem, we see that this is just equal to

$$\left\{k+1 \middle| k+1\right\}$$

□

What if we stack two green stalks together? Now we are left with the state

$$\left\{0, \star \middle| 0, \star\right\}$$

If we consider

$$\left\{0, \star \middle| 0, \star\right\} + \left\{0 \middle| 0\right\}$$

We can see that the first player always wins, as once they move the left $\star$, this leaves the second with $\star + \star = 0$. Therefore, we see

$$\left\{0, \star \middle| 0, \star\right\} \neq \star$$

Definition 17 (Nimber)**.** We define $\star k$ as

$$\left\{0, \star, \star 2, \ldots, \star(k-1) \middle| 0, \star, \star 2, \ldots, \star(k-1)\right\}$$

Note that we cannot simplify this using the simplicity theorem, since $\star \mid\mid 0$. We call the position $\star k$ as the k-th nimber.

We note that this is the same as a k pile of green branches, as any move from this state can go to any of $\star(k-i)$ positions.

Theorem 10 (Nimber Addition)**.**

$$\star n + \star n = 0$$

Proof 13. No matter who plays first, they will move $\star n \to \star(n-k)$. Then the opposite player can do the same thing on the other pile, so in the end the second player makes the last move, winning. □

Theorem 11 (Nimber Addition 2)**.** For $n \neq m$,

$$\star\left(2^n\right) + \star\left(2^m\right) = \star\left(2^n + 2^m\right)$$

Proof 14. We proceed by strong induction.

Suppose that for all b and a where $0 \geq b < a < k$, that $\star\left(2^b\right) + \star\left(2^a\right) = \star\left(2^b + 2^a\right)$. Note that this means the sums of any two nimbers up to 2^k are other nimbers, as you can split apart any nimber into sums of powers of two, cancel pairs, and add the remaining ones together.

We will now consider the sum

$$\star\left(2^a\right)+\star\left(2^k\right)$$

We show that the position $\star\left(2^a\right)+\star\left(2^k\right)+\star\left(2^a+2^k\right)$ is zero.
On the first move, the first player has three options.

1. If the first player moves $\star\left(2^a\right) \to \star x$ for some x, the second player can move $\star\left(2^a+2^k\right) \to \star x+\star\left(2^k\right)$. This sum is a nimber $\star y$ where $y<k$ by induction, so we can perform this move.
2. If the first player moves $\star\left(2^k\right) \to \star x$ for some x, the second player again moves $\star\left(2^a+2^k\right) \to \star x+\star\left(2^a\right)$ (which is possible by the same reasoning as Case 1).
3. If the first player moves in $\star\left(2^a+2^k\right) \to \star x$ for some x, the second player is always able to do one of these two moves (note that these are just case 1 and case 2 but reversed):
   (a) Move $\star\left(2^a\right) \to \star y$ where $\star y+\star\left(2^k\right)=\star x$
   (b) Move $\star\left(2^k\right) \to \star y$ where $\star y+\star\left(2^a\right)=\star x$

Case 1 leaves the first player in a position with value $\star x+\star\left(2^k\right)+\left(\star x+\star\left(2^k\right)\right)=0$. Case 2 leave the first player in a position with $\star\left(2^a\right)+\star x+\left(\star x+\star\left(2^a\right)\right)=0$. Case 3 leaves the first player in either the same outcome as Case 1 or Case 2.

In all three cases, the first player is left in a zero position, so they lose. Therefore,

$$\star\left(2^a\right)+\star\left(2^k\right)=\star\left(2^a+2^k\right)$$

□

Theorem 12 (Nimber Addition 3)**.** For any n, m, we have

$$\star n+\star m=\star(n \oplus m)$$

where $\oplus$ is the bitwise XOR operator.

Proof 15. We can split $\star n$ into powers of two. Let

$$n=n_d n_{d-1} \ldots n_1$$

be the bit representation of n, and likewise for m. Then

$$\star n=\star n_d+\star n_{d-1}+\ldots \star n_1$$

$$\star m=\star m_d+\star m_{d-1}+\ldots \star m_1$$

Now note that if the sum of $\star n$ contains $\star n_i$ and the sum of $\star m$ contains $\star m_i$ where $m_i=n_i=2^k$, then those two values sum to 0. Otherwise, we keep $\star m_i$ or $\star n_i$. Therefore, if we have exactly one 2^k for some k, then the sum will contain $\star 2^k$, otherwise it will not. Therefore, we have that

$$\star n + \star m = \star (n \oplus m)$$

□

3.2 Reversible Moves

We introduce a variant of Nim, called Poker-Nim. The rules are the same, except each player is allowed to keep any stones they remove from the heap. On their turn, instead of removing stones, they may add their stones to an existing heap, or create a new heap.

Theorem 13. Poker Nim is equivalent to regular Nim.

Proof 16. On a player's turn, say they put in x stones into some heap. Then the other player can always take away x stones from that heap, leaving it on the first player's turn again. Therefore, in the end no matter how many stones are placed, this is still equivalent to Nim. □

Definition 18 (Reversible Move)**.** Consider a general position

$$G = \left\{ A, B, C, \ldots \middle| D, E, F, \ldots \right\}$$

We say that Red's move to D is reversible if there exists some D^L where $D^L \geq G$. Likewise, we say that Left's move to A is reversible if there exists some A^R where $A^R \leq G$.

To describe this definition, suppose we start in a position G. You then decide to move from G to H. If your opponent has any option I from H that leaves them in a position at least as good for them as G was, then your move H was reversible. We state that H is reversible through I.

Theorem 14 (Reversible Moves)**.** Consider a position

$$G = \left\{ A, B, C \ldots \middle| X, Y, Z, \ldots \right\}$$

If A is reversible through H, then

$$G = \left\{ H^L, B, C, \ldots \middle| X, Y, Z, \ldots \right\}$$

Likewise, if X is reversible through W, then

$$G = \left\{ A, B, C, \ldots \middle| W^R, Y, Z, \ldots \right\}$$

Proof 17. We only show the proof for Left. The argument for Right is similar.

Let

$$G = \left\{A, B, C, \dots \middle| X, Y, Z, \dots\right\}$$

$$G' = \left\{H^L, B, C, \dots \middle| X, Y, Z, \dots\right\}$$

Suppose that A is reversible through H, meaning $H \leq G$. We wish to show that $G = G'$, which we will show by second player win. Consider the game

$$G - G' = \left\{A, B, C, \dots \middle| X, Y, Z, \dots\right\} - \left\{H^L, B, C, \dots \middle| X, Y, Z\right\}$$

Every move that the first player makes on one component can be copied by the second player on the other, leaving the position zero, except if Left moves $G \to A$ or if Right moves $G' \to H^L$.

- Consider if Left moves $G \to A$. Suppose then that Right moves $A \to H$, leaving the total position $H - G'$ with Left to move.
  - If Left moves $H \to H^L$, then Right can move $G' \to H^L$, leaving the position 0 with Left to move, so Right wins.
  - If Left moves G' to some X, then the total position is $H - X$. Since we know that $H \leq G$, Right wins $H - G$ moving second. This means Right has a winning response to any of Left's moves in $H - G$. Since $H - X$ is one such option, Right has a winning response.

  Therefore, we see that Right wins in this scenario.
- Suppose Right moves $G' \to H^L$. Since we know that $H \leq G$, this means that Left wins $G - H$ playing second. Therefore, any of Right's options in $G - H$ have a winning response. Since $G - H^L$ is one such option, Left wins this scenario.

Therefore, the second player wins both scenarios, so $G = G'$. □

Combining this result with the removal of dominated options allows us to get a single simplest form for any position.

Example. Consider the Nim position

$$\left\{0, \star, \star 2, \star 4, \star 30 \middle| 0, \star, \star 2, \star 4, \star 30\right\}$$

If either player moves to $\star 4$ or $\star 30$, the other player can "reverse" this move by moving to $\star 3$. Moving to $\star 4$ or $\star 30$ is somewhat like the Poker-Nim, and it stalls time. What is special about $\star 3$? Note that it is the first value missing in our set. Every nimber beyond $\star 3$ can simply be reversed by going to $\star 3$.

Definition 19 (Minimum Excludant)**.** The minimum excludant or mex of a set $\{0, 1, \dots\}$ is the first value in $\mathbb{N}$ which is not in the set.

Theorem 15 (Mex Rule)**.** If we have a position $\left\{ \star a, \star b, \star c, \ldots \middle| \star a, \star b, \star c, \ldots \right\}$, then it is equal to

$$\star \left(\text{mex} \left(\{a, b, c, \ldots\} \right) \right)$$

Proof 18. This follows from equality of reversible moves. □

Theorem 16 (Sprague Grundy)**.** Every impartial position is equal to $\star n$ for some n.

Proof 19. We proceed by induction.

For the base case, $0 = \star 0$.

Consider an impartial position

$$G = \left\{ A, B, C, \ldots \middle| A, B, C, \ldots \right\}$$

where all the options are nimbers (by induction). Thus, by the Mex rule, G is also a nimber. □

Chapter 4

Infinitesimals

4.1 Dicotics and Up

Definition 20 (Infinitesimal). We call a value x as infinitesimal if for all positive dyadic fractions d,

$$-d < x < d$$

Definition 21 (Up). We call the game

$$\uparrow = \left\{0, \star, \left\{0, \star \middle| 0\right\} \middle| \star, \left\{0, \star \middle| 0\right\}\right\} = \left\{0 \middle| \star\right\}$$

Proposition 8. $\uparrow$ is positive.

Definition 22 (Down). We call the game $\downarrow$ as the negative of up, or

$$\downarrow = \left\{\star \middle| 0\right\}$$

Definition 23 (Multiples). We notate $\uparrow + \uparrow = \Uparrow$ as double up, $\downarrow + \downarrow = \Downarrow$ as double down. Higher multiples are notated as $n.\uparrow$ and $n.\downarrow$.

Proposition 9. $\uparrow$ and $\downarrow$ are infinitesimal.

Proposition 10. $\uparrow < \Uparrow < 3.\uparrow < \dots$

Proposition 11. $\uparrow - \star$ is a first player win, so $\uparrow$ is confused with $\star$

Theorem 17. For any $n \geq 2$, $\uparrow > \star n$.

Proof 20. Consider the position $G = \uparrow - \star n = \uparrow + \star n$. Since Left has a move to the positive position $\uparrow$ by moving $\star n \to 0$, Left wins going first.

If Right goes first, they can either

- Move $\star n \to \star a$. Then Left can move $\star a \to 0$, so Left wins.
- Right can also move $\uparrow \to \star$, the total position is $\star + \star n$. Since $n > 1$, this position is equal to some nimber other than 0, which is a fuzzy position (first player wins). Since Left is the next player, they win.

Since Left wins, $G > 0$, so $\uparrow > \star n$. □

Definition 24 (Dicotic)**.** A game is considered dicotic if both players have some move. 0 is a dicotic position, and $G = \left\{ G^L \middle| G^R \right\}$ is dicotic if G^L is dicotic and G^R is dicotic.

Theorem 18 (Lawnmower Theorem)**.** Every dicotic position is infinitesimal.

Proof 21. We proceed by induction. Base case: we see that 0 is infinitesimal.

Suppose $G = \left\{ G^L \middle| G^R \right\}$ is dicotic. This means G^L and G^R are dicotic, and we assume them to be infinitesimal by induction.

Consider the position $x - G$, where $x > 0$ is some positive number in canonical form.

- If Right goes first, then either they move to $x^R - G$ or to $x - G^L$ (negative). If Right moves to $x^R - G$, x^R is a positive number, so by induction, Left wins. If Right moves to $x - G^L$, Left wins by induction again.
- If Left goes first, then left moves to $x - G^R$ as G is dicotic. This is a win for Left by induction.

Therefore, $x - G > 0$, so $x > G$. The proof for the negative follows similarly. □

Corollary. Every multiple of Up is infinitesimal.

Definition 25 (Infinitesimally Smaller)**.** If every finite multiple of A is less than B, we write $A \ll B$.

Corollary. $\uparrow \ll 1$

Definition 26 (Tiny and Miny)**.** We call

$$+_x = \left\{0 \middle| \left\{0 \middle| -x\right\}\right\}$$

as "tiny-x". Its negative,

$$-_x = \left\{\left\{x \middle| 0\right\} \middle| 0\right\}$$

is known as "miny-x".

Proposition 12. $+_1 \ll \uparrow$

Proof 22. Let $G = \left\{0 \middle| \left\{0 \middle| -1\right\}\right\}$

Consider the sum

$$n.G + \downarrow$$

We proceed by induction. Base case: $0.G + \downarrow$ is negative, so Right wins it.

Inductive step: Whenever it is Right's turn, they can move from some $G \to \left\{0 \middle| -1\right\}$. Left cannot allow this position to persist and must move $\left\{0 \middle| -1\right\} \to 0$, as if Left moved $\downarrow \to \star$, then Right can move $\left\{0 \middle| -1\right\} \to -1$, leaving the position as $-1 + \star + (n-1).G$. $\star$ is infinitesimal by the Lawnmower theorem, and $(n-1).G$ is infinitesimal by induction, so the sum is negative and Right wins.

Therefore, Left must move $\left\{0 \middle| -1\right\} \to 0$. This leaves the position at $(n-1).G + \downarrow$. By induction, this is negative, so Right wins. Since Right wins no matter what, $n.G < \uparrow$, so

$$G \ll \uparrow$$

□

Theorem 19. Let $x > y \geq 0$. Then

$$+_x \ll +_y$$

Proof 23. Consider the sum $n.+_x + -_y$, for any $n > 1$. Whenever it is Right's turn, they can move from some $+_x \to \left\{0 \middle| -x\right\}$. Left cannot allow this position to persist and must move it to 0, because if Right were allowed to move in it, they could take it to $-x$. Even if Left converted $-_y$ to y, the total position would be $-x + y + z$ for an infinitesimal z, which is still negative since $x > y$. Thus Left must move $\left\{0 \middle| -x\right\} \to 0$. Right can repeat this process with every copy of $+_x$, until the position left is $-_y$, at which point Right wins.

Since Right wins no matter what, we see that $n.+_x < +_y$. Therefore,

$$+_x \ll +_y$$

□

Theorem 20 (Extremely Tiny)**.** For any positive G, there exists some x such that $+_x \leq G$.

Proof 24. Choose n such that for all G^{RR}, $n > G^{RR}$. Such an n must exist by the Archimedian Principle for Integers. Consider the position

$$H = G + -_n$$

If Right moves, they must move in G since their move in $-_n$ loses since G is positive. So Right must move to

$$G^R + -_n$$

Left here can move $-_n \to \left\{n \middle| 0\right\}$. Right cannot avoid this threat as $n > G^{RR}$, so they must move $\left\{n \middle| 0\right\} \to 0$. Therefore, the final position G^R $||> 0$.

Therefore, H must be greater than or equal to 0, so

$$+_x \leq G$$

□

Our current view of the infinitesimals is as follows. We know that

$$0 \ldots +_n \ll +_{n-1} \ll \ldots \ll +_1 \ll \uparrow < \Uparrow < 3.\uparrow < \ldots < 1$$

However, we are missing quite a few other infinitesimals. We now introduce a new operator which will let us look in between.

4.2 Ordinal Sums and Flower Gardens

Definition 27 (Ordinal Sum)**.** We define

$$G : H = \left\{G^L, G : H^L \middle| G^R, G : H^R\right\}$$

as the ordinal sum of G and H. We can think of this as putting the Hackenbush state H "on top" of G, so any move in G erases H.

Proposition 13. $\star a : \star b = \star(a + b)$

However, the ordinal sum does not satisfy all our nice properties. For example,
$0 : 1 = 1$, $\left\{0\middle|0\right\} = 0$, but $\left\{\star\middle|\star\right\} : 1 = \uparrow$.

Definition 28 (Flower). A flower is a position of the form $\star m : n$, where n is an integer and m is a positive integer.

Definition 29 (Flower Color). A flower is green if $n = 0$, red if $n < 0$, and blue if $n > 0$.

Definition 30 (Flower Garden). A flower garden is a sum of flowers.

Figuring out who wins a flower garden is hard. In fact, it is an open problem. However, we can find out who wins for some flower gardens.

Definition 31 (Weight). The weight of a flower garden, denoted $w(G)$, is the difference between the count of the blue flowers and the count of red flowers.

Theorem 21 (Two Ahead Rule).

(a) If $w(G) \geq 1$, then $G \mathrel{||\!>} 0$.

(b) If $w(G) \geq 2$, then $G > 0$.

4.3 Uptimals

Definition 32 (Up-nth). We define the n-th uptimal

$$\uparrow^n = (\star : n) - (\star : (n-1))$$

Definition 33 (Uptimal ??? (No name)).

$$\uparrow^{[n]} = (\star : n) - \star = \uparrow + \uparrow^2 + \uparrow^3 + \ldots + \uparrow^n$$

Definition 34 (Uptimal). Any position of the form

$$k_1. \uparrow + k_2. \uparrow^2 + \ldots + k_n. \uparrow^n$$

where k_i can be any integer.

Theorem 22. For all $n \geq 1$,

$$\uparrow^{n+1} \ll \uparrow^{n}$$

Theorem 23. For all $n \geq 1$, and for all k,

$$k.\uparrow^{2} || \star n$$

This means that any uptimal that doesn't include $\uparrow$ or $\downarrow$ is confused with every nonzero nimber, as it must contain some

$$\uparrow^{n}, n \geq 2$$

Theorem 24. For all $n \geq 1$,

$$+_1 \ll \uparrow^{n}$$

Chapter 5

Transfinite Games

5.1 Transfinite Games

So far we've only discussed games which are short, or where the game must end at some finite point. However, games can similarly be extended to long games.

We now permit G^L and G^R to be infinite sets of options, which leads to the rise of games such as

$$\omega = \left\{0, 1, 2, 3, \ldots \middle| \right\}$$

and

$$\star\omega = \left\{0, \star, \star 2, \star 3, \ldots \middle| 0, \star, \star 2, \star 3, \ldots \right\}$$

We first note that the Fundamental Theorem of Combinatorial Game Theory still holds for long games. We can repeat the proof from before, this time using transfinite induction for the inductive step.

Most statements made in short game theory still hold for long game theory.

Through these, we can now construct the reals. For example,

$$G = \left\{0, \frac{1}{4}, \frac{5}{16}, \ldots \middle| \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \ldots \right\}$$

is equivalent to $\frac{1}{3}$. We can see that the definitions of G for real values form Dedekind cuts.

Definition 35 (The Reals)**.** For $x \in \mathbb{R}$, we construct the game $\left\{X^L \middle| X^R\right\}$ where $X^L = \{q \in \mathbb{D} | q < x\}$ and $X^R = \{q \in \mathbb{D} | q > x\}$.

However, the structure of games allows us to construct a multitude more games than just the reals. In fact, we can construct the **surreals**, which is the largest ordered field.

Definition 36 (Surreal Number)**.** A long game x is a surreal number if $y^L < y^R$ for every subposition y of x and every y^L and y^R.

Example. Consider the game

$$\eta = \left\{0 \middle| 1, \frac{1}{2}, \frac{1}{4}, \dots\right\}$$

We see that $\eta < x$ for every positive $x \in \mathbb{D}$, but also $\eta > 0$, since $\eta^L = 0$. So η is infinitesimal.

Example (Some ordinals and surreal numbers).

$$\begin{aligned}
\omega - 1 &= \left\{0, 1, 2, \dots \middle| \omega\right\} \\
\frac{1}{2}\omega &= \left\{0, 1, 2, \dots \middle| \omega, \omega - 1, \omega - 2, \dots\right\} \\
\frac{1}{4}\omega &= \left\{0, 1, 2, \dots \middle| \frac{1}{2}\omega, \frac{1}{2}\omega - 1, \frac{1}{2}\omega - 2, \dots\right\} \\
\sqrt{\omega} &= \left\{0, 1, 2, \dots \middle| \omega, \frac{1}{2}\omega, \frac{1}{4}\omega, \dots\right\} \\
\frac{1}{\omega} &= \left\{0 \middle| 1, \frac{1}{2}, \frac{1}{4}, \dots\right\}
\end{aligned}$$

One interesting thing we see is that

$$\frac{1}{\omega} = \left\{0 \middle| 1, \frac{1}{2}, \frac{1}{4}, \dots\right\} \ll x$$

for all real x. However, we can also see that

$$\uparrow \ll \frac{1}{\omega}$$

through a generalization of the Lawnmower theorem. Now the surreal number structure also forms a field structure with the addition of a multiplication and division operations. However, these are extremely complicated (involving recursive definitions), which will not be covered here.

Chapter 6

Loopy Games

6.1 Loopy Games

For all games previously discussed, it was impossible to double back on oneself. If we relax this condition, we encounter a whole new host of games. To distinguish standard games, we will call such games **loopfree** games.

Definition 37 (Loopy)**.** A game G is loopy if it contains a cycle in its graph representation.

Example (On and Off)**.**

$$\text{on} = \left\{ \text{on} \middle| \right\}$$

$$\text{off} = \left\{ \middle| \text{off} \right\}$$

Example (Dud)**.**

$$\text{dud} = \left\{ \text{dud} \middle| \text{dud} \right\}$$

Example (Tis and Tism)**.**

$$\text{tis} = \left\{ \text{tisn} \middle| \right\}$$

$$\text{tisn} = \left\{ \middle| \text{tis} \right\}$$

Example (Over and Under)**.**

$$\text{over} = \left\{ 0 \middle| \text{over} \right\}$$

$$\text{under} = \left\{\text{under}\middle|0\right\}$$

6.2 Draws

Now note that the Fundamental Theorem of Combinatorial Game Theory does not hold. For example, in the **dud** game, Left and Right can keep playing forever. Therefore, this gives us a new host of possible results summarized in this table:

Table 6.1: New Outcome Classes

		Left goes first		
		Left wins	Draw	Right Wins
Right goes first	Left wins	$\mathcal{L}$	$\hat{\mathcal{P}}$	$\mathcal{P}$
	Draw	$\hat{\mathcal{N}}$	$\hat{\mathcal{D}}$	$\check{\mathcal{P}}$
	Right wins	$\mathcal{N}$	$\check{\mathcal{N}}$	$\mathcal{R}$

The hat and check outcome classes can be thought of as outcome classes which are slightly more favorable towards Left and Right. For example, if Left wins going first but Draws going second, this is favorable towards Left, but not in the $\mathcal{L}$ class.

What is a draw? If neither player wants to lose but cannot force a win, they can force an infinite stalemate by repeatedly playing moves. For example, **on** + **off** = **dud** is a game in the outcome class $\mathcal{D}$.

These outcome classes also admit a partial ordering.

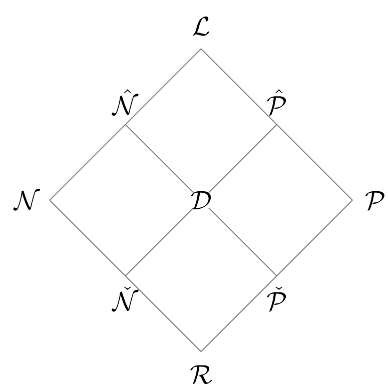

Figure 6.1: Partial Ordering

Note that things higher up on this chart are better for Left. This means we can now define orderings. For example, $\hat{\mathcal{N}} > \mathcal{D}$, but $\hat{\mathcal{N}} || \mathcal{P}$.

Importantly, note that our previous method of comparing games also no longer works. Previously, we had that $G = H$ when $G - H = 0$. However, we see that $\textbf{on} - \textbf{on} = \textbf{on} + \textbf{off} = \textbf{dud} \neq 0$.

Therefore, we must go back to the outcome class definition of equality:

Definition 38 (Comparison). $G \geq H$ if for all (possibly loopy) X,

$$o(G+X) \geq o(H+X)$$

A similar definition can be provided for $=$, $\leq$, $||$.

- If $G \geq H$, then Left can survive (avoid losing) $G - H$ when going second.
- If Left can win $G - H$ going second, then $G \geq H$.

This asymmetry creates problems with Loopy games.

Definition 39 (Stopper). A loopy game G is a stopper if play on G is always finite.

Example. **on** is a stopper as it ends in one move, while **dud** is not a stopper as it will never end.

6.3 Stoppers and Sides

We can now define biased outcome classes. Instead of a singular outcome class, we can instead consider the outcome classes $\hat{o}(X)$ and $\check{o}(X)$.

Definition 40 (Biased Outcome Classes). Let $\hat{o}(X)$ be the outcome class of the game X where we consider a draw to be a win for Left, and $\check{o}(X)$ the outcome class of the game X where we consider a draw to be a win for Right.

Note that the biased outcome classes are one of the four original outcome classes. Together, we can see that the biased outcome classes determine the full 9 way outcome class.

Definition 41 (Biased (in)equality). $G \hat{\geq} H$ if for all (possibly loopy) X,

$$\hat{o}(G+X) \geq \hat{o}(H+X)$$

A similar definition can be provided for $\hat{=}$, $\hat{\leq}$, $\hat{||}$, and similarly for $\check{\geq}$.

Definition 42 (Sides). Let G be a loopy game. If there exist stoppers S and T such that $G \hat{=} S$ and $G \check{=} T$, we say that G is **stopper-sided**, has S as its **onside** and T as its **offside**. We denote this as $G = S\&T$.

An important theorem establishes that S and T are unique if they exist. However, we will not cover this.

Example. We can see that $\hat{o}(\mathbf{on}) = \hat{o}(\mathbf{on}) = \mathcal{L}$, and so by symmetry,

$$\mathbf{dud} = \mathbf{on}\&\mathbf{off}$$